OCR
RECOGNISING ACHIEVEMENT

ADVANCED GCE

MATHEMATICS

4727

Further Pure Mathematics 3

Candidates answer on the answer booklet.

OCR supplied materials:
- 8 page answer booklet (sent with general stationery)
- List of Formulae (MF1)

Other materials required:
- Scientific or graphical calculator

Friday 28 January 2011
Morning

Duration: 1 hour 30 minutes

INSTRUCTIONS TO CANDIDATES

- Write your name, centre number and candidate number in the spaces provided on the answer booklet. Please write clearly and in capital letters.
- Use black ink. Pencil may be used for graphs and diagrams only.
- Read each question carefully. Make sure you know what you have to do before starting your answer.
- Answer **all** the questions.
- Do **not** write in the bar codes.
- Give non-exact numerical answers correct to 3 significant figures unless a different degree of accuracy is specified in the question or is clearly appropriate.
- You are permitted to use a scientific or graphical calculator in this paper.

INFORMATION FOR CANDIDATES

- The number of marks is given in brackets **[ ]** at the end of each question or part question.
- **You are reminded of the need for clear presentation in your answers.**
- The total number of marks for this paper is **72**.
- This document consists of **4** pages. Any blank pages are indicated.

1 **(i)** Find the general solution of the differential equation

$$\frac{dy}{dx} + xy = xe^{\frac{1}{2}x^2},$$

giving your answer in the form $y = f(x)$. **[4]**

(ii) Find the particular solution for which $y = 1$ when $x = 0$. **[2]**

2 Two intersecting lines, lying in a plane p, have equations

$$\frac{x-1}{2} = \frac{y-3}{1} = \frac{z-4}{-3} \quad \text{and} \quad \frac{x-1}{-1} = \frac{y-3}{2} = \frac{z-4}{4}.$$

(i) Obtain the equation of p in the form $2x - y + z = 3$. **[3]**

(ii) Plane q has equation $2x - y + z = 21$. Find the distance between p and q. **[3]**

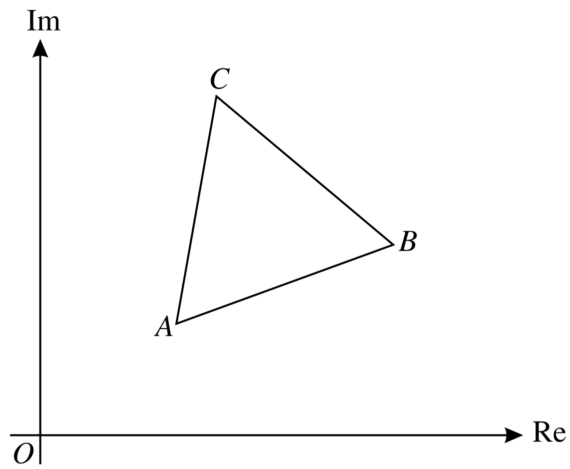
3 **(i)** Express $\sin\theta$ in terms of $e^{i\theta}$ and $e^{-i\theta}$ and show that

$$\sin^4\theta \equiv \tfrac{1}{8}(\cos 4\theta - 4\cos 2\theta + 3).$$ **[4]**

(ii) Hence find the exact value of $\int_0^{\frac{1}{6}\pi} \sin^4\theta \, d\theta$. **[4]**

4 The cube roots of 1 are denoted by 1, ω and ω^2, where the imaginary part of ω is positive.

(i) Show that $1 + \omega + \omega^2 = 0$. **[2]**



In the diagram, ABC is an equilateral triangle, labelled anticlockwise. The points A, B and C represent the complex numbers z_1, z_2 and z_3 respectively.

(ii) State the geometrical effect of multiplication by ω and hence explain why $z_1 - z_3 = \omega(z_3 - z_2)$. **[4]**

(iii) Hence show that $z_1 + \omega z_2 + \omega^2 z_3 = 0$. **[2]**

5 **(i)** Find the general solution of the differential equation

$$3\frac{d^2y}{dx^2} + 5\frac{dy}{dx} - 2y = -2x + 13.$$ **[7]**

(ii) Find the particular solution for which $y = -\frac{7}{2}$ and $\frac{dy}{dx} = 0$ when $x = 0$. **[5]**

(iii) Write down the function to which y approximates when x is large and positive. **[1]**

6 Q is a multiplicative group of order 12.

(i) Two elements of Q are a and r. It is given that r has order 6 and that $a^2 = r^3$. Find the orders of the elements a, a^2, a^3 and r^2. **[4]**

The table below shows the number of elements of Q with each possible order.

Order of element	1	2	3	4	6
Number of elements	1	1	2	6	2

G and H are the non-cyclic groups of order 4 and 6 respectively.

(ii) Construct two tables, similar to the one above, to show the number of elements with each possible order for the groups G and H. Hence explain why there are no non-cyclic proper subgroups of Q. **[5]**

7 Three planes Π_1, Π_2 and Π_3 have equations

$$\mathbf{r}.(\mathbf{i} + \mathbf{j} - 2\mathbf{k}) = 5, \qquad \mathbf{r}.(\mathbf{i} - \mathbf{j} + 3\mathbf{k}) = 6, \qquad \mathbf{r}.(\mathbf{i} + 5\mathbf{j} - 12\mathbf{k}) = 12,$$

respectively. Planes Π_1 and Π_2 intersect in a line l; planes Π_2 and Π_3 intersect in a line m.

(i) Show that l and m are in the same direction. **[5]**

(ii) Write down what you can deduce about the line of intersection of planes Π_1 and Π_3. **[1]**

(iii) By considering the cartesian equations of Π_1, Π_2 and Π_3, or otherwise, determine whether or not the three planes have a common line of intersection. **[4]**

[Question 8 is printed overleaf.]

8 The operation $*$ is defined on the elements (x, y), where $x, y \in \mathbb{R}$, by

$$(a, b) * (c, d) = (ac, ad + b).$$

It is given that the identity element is $(1, 0)$.

(i) Prove that $*$ is associative. **[3]**

(ii) Find all the elements which commute with $(1, 1)$. **[3]**

(iii) It is given that the particular element (m, n) has an inverse denoted by (p, q), where

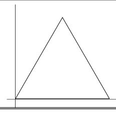
$$(m, n) * (p, q) = (p, q) * (m, n) = (1, 0).$$

Find (p, q) in terms of m and n. **[2]**

(iv) Find all self-inverse elements. **[3]**

(v) Give a reason why the elements (x, y), under the operation $*$, do not form a group. **[1]**

1	(i)	Integrating factor. $e^{\int x\,dx} = e^{\frac{1}{2}x^2}$	B1		For correct IF
		$\Rightarrow \frac{d}{dx}\left(y e^{\frac{1}{2}x^2}\right) = x e^{x^2}$	M1		For $\frac{d}{dx}(y.\text{their IF}) = x e^{\frac{1}{2}x^2}.\text{their IF}$
		$\Rightarrow y e^{\frac{1}{2}x^2} = \frac{1}{2}e^{x^2}(+c)$	A1		For correct integration both sides
		$\Rightarrow y = e^{-\frac{1}{2}x^2}\left(\frac{1}{2}e^{x^2} + c\right) = \frac{1}{2}e^{\frac{1}{2}x^2} + c\,e^{-\frac{1}{2}x^2}$	A1	4	For correct solution **AEF** as $y = f(x)$
	(ii)	$(0,\ 1) \Rightarrow c = \frac{1}{2}$	M1		For substituting (0, 1) into their GS, solving for *c* and obtaining a solution of the DE
		$\Rightarrow y = \frac{1}{2}\left(e^{\frac{1}{2}x^2} + e^{-\frac{1}{2}x^2}\right)$	A1	2	For correct solution **AEF** Allow $y = \cosh\left(\frac{1}{2}x^2\right)$
				6	
2	(i)	$\mathbf{n} = [2, 1, -3] \times [-1, 2, 4]$	M1		For using × of direction vectors
		$= [10, -5, 5] = k[2, -1, 1]$	A1		For correct **n**
		$(1, 3, 4) \Rightarrow 2x - y + z = 3$	A1	3	For substituting $(1, 3, 4)$ and obtaining **AG** (Verification only M0)
	(ii)	METHOD 1 distance = $\frac{21-3}{\lvert\mathbf{n}\rvert}$ *OR* $\frac{\lvert[1, 3, 4].[2, -1, 1] - 21\rvert}{\lvert\mathbf{n}\rvert}$	M1		For 21 – 3 *OR* $[1, 3, 4].[2, -1, 1] - 21$ *OR* $\lvert([1, 3, 4] - [a, b, c]).[2, -1, 1]\rvert$ soi
		OR $\frac{\lvert([1, 3, 4] - [a, b, c]).[2, -1, 1]\rvert}{\lvert\mathbf{n}\rvert}$ where (a, b, c) is on *q*	B1		For $\lvert\mathbf{n}\rvert = \sqrt{6}$ soi
		$= \frac{18}{\sqrt{6}} = 3\sqrt{6}$	A1	3	For correct distance **AEF**
		METHOD 2 $[1+2t, 3-t, 4+t]$ on *q*	M1		For forming and solving an equation in *t*
		$\Rightarrow 2(1+2t) - (3-t) + (4+t) = 21 \Rightarrow t = 3$	B1		For $\lvert\mathbf{n}\rvert = \sqrt{6}$ soi
		$\Rightarrow$ distance $= 3\lvert\mathbf{n}\rvert = 3\sqrt{6}$	A1		For correct distance **AEF**
		METHOD 3 As Method 2 to $t = 3 \Rightarrow (7, 0, 7)$ on *q*	M1*		For finding point where normal meets *q*
		distance from $(1, 3, 4)$	M1 (*dep)		For finding distance from $(1, 3, 4)$
		$= \sqrt{(7-1)^2 + (0-3)^2 + (7-4)^2} = \sqrt{54} = 3\sqrt{6}$	A1		For correct distance **AEF**
				6	
3	(i)	$\sin\theta = \frac{1}{2i}\left(e^{i\theta} - e^{-i\theta}\right)$	B1		*z* or $e^{i\theta}$ may be used throughout For correct expression for $\sin\theta$ soi
		$\sin^4\theta = \frac{1}{16}\left(z^4 - 4z^2 + 6 - 4z^{-2} + z^{-4}\right)$	M1		For expanding $\left(e^{i\theta} - e^{-i\theta}\right)^4$ (with at least 3 terms and 1 binomial coefficient)
		$\Rightarrow \sin^4\theta = \frac{1}{16}(2\cos 4\theta - 8\cos 2\theta + 6)$	M1		For grouping terms and using multiple angles
		$\Rightarrow \sin^4\theta = \frac{1}{8}(\cos 4\theta - 4\cos 2\theta + 3)$	A1	4	For answer obtained correctly **AG**
	(ii)	$\int_0^{\frac{1}{6}\pi} \sin^4\theta\,d\theta = \frac{1}{8}\left[\frac{1}{4}\sin 4\theta - 2\sin 2\theta + 3\theta\right]_0^{\frac{1}{6}\pi}$	M1		For integrating **(i)** to $A\sin 4\theta + B\sin 2\theta + C\theta$
			A1		For correct integration
		$= \frac{1}{8}\left(\frac{1}{8}\sqrt{3} - \sqrt{3} + \frac{1}{2}\pi\right) = \frac{1}{64}\left(4\pi - 7\sqrt{3}\right)$	M1		For completing integration and substituting limits
			A1	4	For correct answer **AEF**(exact)
				8	

4 (i)	*EITHER* $1+\omega+\omega^2$ $=$ sum of roots of $(z^3-1=0)=0$	M1 A1 **2**	For result shown by any correct method **AG**
	OR $\omega^3=1 \Rightarrow (\omega-1)(\omega^2+\omega+1)=0$ $\Rightarrow 1+\omega+\omega^2=0$ (for $\omega\neq 1$)		
	OR sum of G.P. $1+\omega+\omega^2=\frac{1-\omega^3}{1-\omega}\left(=\frac{0}{1-\omega}\right)=0$		
	OR  shown on Argand diagram or explained in terms of vectors		
	OR $1+\text{cis}\,\frac{2}{3}\pi+\text{cis}\,\frac{4}{3}\pi=1+\left(-\frac{1}{2}+\frac{\sqrt{3}}{2}\text{i}\right)+\left(-\frac{1}{2}-\frac{\sqrt{3}}{2}\text{i}\right)=0$		
(ii)	Multiplication by $\omega \Rightarrow$ rotation through $\frac{2}{3}\pi$ ↺	B1	For correct interpretation of × by ω (allow 120° and omission of, or error in, ↺)
	$z_1-z_3=\overrightarrow{CA}$, $z_3-z_2=\overrightarrow{BC}$	B1	For identification of vectors soi (ignore direction errors)
	$\overrightarrow{BC}$ rotates through $\frac{2}{3}\pi$ to direction of $\overrightarrow{CA}$	M1	For linking BC and CA by rotation of $\frac{2}{3}\pi$ *OR* ω
	Δ *ABC* has $BC = CA$, hence result	A1 **4**	For stating equal magnitudes $\Rightarrow$ **AG**
(iii)	**(ii)** $\Rightarrow z_1+\omega z_2-(1+\omega)z_3=0$	M1	For using $1+\omega+\omega^2=0$ in **(ii)**
	$1+\omega+\omega^2=0 \Rightarrow z_1+\omega z_2+\omega^2 z_3=0$	A1 **2**	For obtaining **AG**
		8	
5 (i)	Aux. equation $3m^2+5m-2\ (=0)$	M1	For correct auxiliary equation seen and solution attempted
	$\Rightarrow m=\frac{1}{3}, -2$	A1	For correct roots
	CF $(y=)\ A\text{e}^{\frac{1}{3}x}+B\text{e}^{-2x}$	A1√	For correct CF f.t. from m with 2 arbitrary constants
	PI $(y=)\ px+q \Rightarrow 5p-2(px+q)=-2x+13$	M1	For stating and substituting PI of correct form
	$\Rightarrow p=1,\quad q=-4$	A1 A1	For correct value of p, and of q
	GS $(y=)\ A\text{e}^{\frac{1}{3}x}+B\text{e}^{-2x}+x-4$	B1√ **7**	For GS f.t. from their CF+PI with 2 arbitrary constants in CF and none in PI
(ii)	$\left(0,-\frac{7}{2}\right) \Rightarrow A+B=\frac{1}{2}$	M1	For substituting $\left(0,-\frac{7}{2}\right)$ in their GS and obtaining an equation in A and B
	$y'=\frac{1}{3}A\text{e}^{\frac{1}{3}x}-2B\text{e}^{-2x}+1$, $(0,0)\Rightarrow A-6B=-3$	M1	For finding y', substituting $(0, 0)$ and obtaining an equation in A and B
		M1	For solving their 2 equations in A and B
	$\Rightarrow A=0,\ B=\frac{1}{2}$	A1	For correct A and B **CAO**
	$\Rightarrow (y=)\ \frac{1}{2}\text{e}^{-2x}+x-4$	B1√ **5**	For correct solution f.t. with their A and B in their GS
(iii)	x large $\Rightarrow (y=)\ x-4$	B1√ **1**	For correct equation or function (allow ≈ and →) **WWW** f.t. from **(ii)** if valid
		13	

Question		Answer	Marks		Guidance
6	**(i)**	$a^4 = r^6 = e \Rightarrow a$ has order 4, a^2 has order 2	M1		For considering powers of a
		$\left(a^3\right)^4 = a^{12} = e \Rightarrow a^3$ has order 4	A1 A1		For order of any one of $a,\ a^2, a^3$ correct For all correct
		$\left(r^2\right)^3 = e \Rightarrow r^2$ has order 3	B1	**4**	For order of r^2 correct
	(ii)	G order 4 Order of element: 1, 2, (4) Number of elements: 1, 3, (0)	M1		For top line in either table Allow inclusion of 4 and 6 respectively (and other orders if 0 appears below)
		H order 6 Order of element: 1, 2, 3, (6) Number of elements: 1, 3, 2, (0)	A1 A1		For order 4 table For order 6 table
		G and H are the only non-cyclic groups of order which divides 12	B1		For stating that only G and H need be considered **AEF**
		Q has 1 element of order 2, G and H have 3, so no non-cyclic subgroups in Q	B1	**5**	For argument completed by elements of order 2 **AG** **SR** Allow equivalent arguments for B1 B1
			9		
7	**(i)**	$[1, 1, -2] \times [1, -1, 3] = (\pm)[1, -5, -2]$	M1 A1		For using × of direction vectors For correct direction
		$[1, -1, 3] \times [1, 5, -12] = (\pm)[-3, 15, 6]$	M1 A1		For using × of direction vectors For correct direction
		$[-3, 15, 6] = k\,[1, -5, -2] \Rightarrow$ parallel	A1	**5**	For argument completed **AG** ($k = -3$ not essential)
	(ii)	Line of intersection is parallel to l and m	B1	**1**	For correct statement
	(iii)	METHOD 1			
		$\left.\begin{matrix} x+y-2z=5 \\ x-y+3z=6 \end{matrix}\right\}$ e.g. $z=0 \Rightarrow \left(\frac{11}{2}, -\frac{1}{2}, 0\right)$ on l	M1 A1		For attempt to find points on 2 lines For a correct point on one line
		$\left.\begin{matrix} x-y+3z=6 \\ x+5y-12z=12 \end{matrix}\right\}$ e.g. $z=0 \Rightarrow (7,\ 1,\ 0)$ on m	A1		For a correct point on another line
		$\left.\begin{matrix} x+y-2z=5 \\ x+5y-12z=12 \end{matrix}\right\}$ e.g. $z=0 \Rightarrow \left(\frac{13}{4}, \frac{7}{4}, 0\right)$ on l_3			
		Different points $\Rightarrow$ no common line of intersection	A1	**4**	For correct answer
		METHOD 2			
		$\left.\begin{matrix} x+y-2z=5 \\ x-y+3z=6 \end{matrix}\right\}$ e.g. $\Rightarrow z = 11-2x,\ y = 27-5x$	M1 A1		For finding (e.g.) y and z in terms of x *OR* eliminating one variable For correct expressions *OR* equations
		LHS of eqn 3 = $x + (135-25x) - (132-24x) = 3 \neq 12$	A1		For obtaining a contradiction from 3rd equation
		$\Rightarrow$ no common line of intersection	A1		For correct answer
		METHOD 3			
		LHS $\Pi_3 = 3\Pi_1 - 2\Pi_2$	M2		For attempt to link 3 equations
		RHS $3\times5 - 2\times6 = 3 \neq 12$	A1		For obtaining a contradiction
		$\Rightarrow$ no common line of intersection	A1		For correct answer
		SR Variations on all methods may gain full credit			**SR** f.t. may be allowed from relevant working
			10		

8	(i)	$((a,b)*(c,d))*(e,f)=(ac,\ ad+b)*(e,f)$	M1	For 3 distinct elements bracketed and attempt to expand
		$=(ace,\ acf+ad+b)$	A1	For correct expression
		$(a,b)*((c,d)*(e,f))=(a,b)*(ce,\ cf+d)$		
		$=(ace,\ acf+ad+b)$	A1 **3**	For correct expression again
	(ii)	$(a,b)*(1,1)=(a,\ a+b)$, $(1,1)*(a,b)=(a,\ b+1)$	M1	For combining both ways round
		$a+b=b+1 \Rightarrow a=1$	M1	For equating components (allow from incorrect pairs)
		$\Rightarrow (1,b)\ \forall\ b$	A1 **3**	For correct elements **AEF**
	(iii)	$(mp,\ mq+n)$ *OR* $(pm,\ pn+q)=(1,0)$	M1	For either element on LHS
		$\Rightarrow (p,q)=\left(\frac{1}{m},-\frac{n}{m}\right)$	A1 **2**	For correct inverse
	(iv)	$(a,b)*(a,b)=(a^2,\ ab+b)=(1,0)$ *OR* $(a,b)=\left(\frac{1}{a},-\frac{b}{a}\right) \Rightarrow a^2=1,\ ab=-b$	M1	For attempt to find self-inverses
		$\Rightarrow$ self-inverse elements $(1,0)$ and $(-1,b)\ \forall\ b$	B1 A1 **3**	For $(1,0)$. For $(-1,b)$ **AEF**
	(v)	$(0,y)$ has no inverse for any $y \Rightarrow$ not a group	B1 **1**	For stating any one element with no inverse. Allow $x \neq 0$ required, provided reference to inverse is made "Some elements have no inverse" B0
			12	